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TOPOLOGY AND ITS APPLICATIONS

# Solutions to two problems of J.D. Lawson and M. Mislove ☆

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**Abstract**

The problem of characterizing those $L$ for which $[X \to L]$ is a continuous DCPO for all core compact spaces is solved. Also, the problem of coincidence of the Isbell topology with the Scott topology on $[X \to L]$ is solved.

*Keywords:* Continuous DCPO; L-Domain; Core compact space; Isbell topology

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## 1. Introduction and preliminaries

Let $X$ be a topological space and $L$ a DCPO (i.e., $L$ is a partially ordered set in which every directed set has the supremum) equipped with the Scott topology $\sigma(L)$. Write $[X \to L]$ for the set of all continuous mappings from $X$ to $L$; then the set $[X \to L]$ with the pointwise order is again a DCPO. Recently, in [1] Lawson and Mislove posed the following problems:

**Problem A.** Characterize those $L$ for which $[X \to L]$ is a continuous DCPO for all core compact spaces $X$. A likely candidate is the class of all continuous L-Domains $L$. Does one get the same answer if one restricts to the core compact spaces which are also compact?

**Problem B.** Let $X$ be a core compact space and $L$ be a DCPO equipped with the Scott topology. Under what conditions on $L$ do the Isbell and Scott topologies on $[X \to L]$ agree?

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The aim of this paper is to give a complete solution to Problem A and a partial solution to Problem B. Firstly, we point out that the last question in Problem A has a negative answer and hence Problem A is solved by means of the following theorem.

**Theorem A.** *For a DCPO $L$, the following conclusions are equivalent:*
(1) *$L$ is a continuous L-Domain;*
(2) *$[X \to L]$ is a continuous L-Domain for all core compact spaces $X$;*
(3) *$[X \to L]$ is a continuous DCPO for all compact spaces $X$.*

Then two examples are given to show that Problem B has no positive answer even for continuous L-Domains with least element and bounded complete continuous DCPOs without least element. Finally, the following conclusion is proved and hence Problem B is solved.

**Theorem B.** *Let $L$ be a continuous L-Domain with least element $O_L$. Then the Isbell and Scott topologies on $[X \to L]$ agree for all core compact spaces $X$ if and only if $L$ is a bounded complete DCPO.*

For a DCPO $L$ and $x, y \in L$, we say that $x$ is way below $y$ ($x \ll y$), if given a directed set $D \subseteq L$ with $y \leqslant \bigvee D$ then $x \leqslant d$ for some $d \in D$. A DCPO $L$ is said to be continuous, if for all $x \in L$ the set $\Downarrow x = \{y \in L\colon\ y \ll x\}$ is directed and $x = \bigvee \Downarrow x$. If $L$ is simultaneously a complete lattice and a continuous DCPO, then $L$ is called a continuous lattice.

The following facts are well-known and often used in the present paper.

**Theorem 1.1** [2]. *In any continuous DCPO $L$, we have the following conclusions:*
(1) *$x \ll y$ implies there is a $z \in L$ such that $x \ll z \ll y$;*
(2) *$\{\Uparrow x = \{y \in L\colon\ y \gg x\}\colon\ x \in L\}$ is a base of the Scott topology $\sigma(L)$;*
(3) *the Scott topology $\sigma(L)$ is a continuous lattice.*

A DCPO $L$ is a L-Domain if for all $x \in L$ the set $\downarrow x = \{y \in L\colon\ y \leqslant x\}$ is a complete lattice; $L$ is called bounded complete if $L$ is a DCPO and each nonempty subset with upper bound has a supremum. Obviously, a bounded complete DCPO is also a L-Domain if $\downarrow x$ has a least element for each $x \in L$, but the reverse is not true.

We say that a topological $X$ is core compact if the lattice $\Omega(X)$ of its open sets is a continuous lattice.

## 2. Solution to Problem A

In the following, for a DCPO $L$ and $x \in L$ we write $\bigvee_x$ for the supremum operation in $\downarrow x$ and $O_x$ for the least element of $\downarrow x$ if it exists, and $\mathbb{N}$ stands for the set of natural numbers.

**Lemma 2.1.** *In any L-Domain $L$, one has the following statements for each $x \in L$:*

*(1) $O_x$ is a minimal element of $L$ and hence $O_x \ll O_x$;*

*(2) if $\{a_i\colon i \in \Delta\} \subseteq L$ with $\bigvee_{i\in\Delta} a_i \geqslant O_x$, then $O_{a_i} = O_x$ for all $i \in \Delta$.*

**Proof.** It directly follows from the definition of $O_x$. □

For a core compact space $X$ and a continuous DCPO $L$ with least element $O_L$, we can easily show, as in [1], $\bigvee \Downarrow f = f$ for each $f \in [X \to L]$ by constructing mappings that take some constant value on an open set of $X$ and value $O_L$ otherwise. For a general continuous DCPO, it seems more difficult to show this. The following construction is essential.

For $A \subseteq X$, $a \in L$ and $f \in [X \to L]$, define a mapping

$$F(a, A, f)\colon X \to L \quad \text{by } F(a, A, f)(x) = \begin{cases} a, & x \in A, \\ O_{f(x)}, & x \notin A, \end{cases}$$

and write $f_{\langle\langle a\rangle\rangle} = f^{-1}(\Uparrow a)$ for short.

**Lemma 2.2.** *Let $L$ be a continuous L-Domain, $a \in L$ and $f \in [X \to L]$. Then*

$$F(a, U, f) \in [X \to L]$$

*if $U \in \Omega(X)$ and $U \subseteq f_{\langle\langle a\rangle\rangle}$.*

**Proof.** Take an arbitrary $b \in L$ and note that $f_{\langle\langle a\rangle\rangle} \cap f_{\langle\langle b\rangle\rangle} = \emptyset$ if $O_b \neq O_a$. Then $f_{\langle\langle b\rangle\rangle} \subseteq X \setminus U$ as $U \subseteq f_{\langle\langle a\rangle\rangle}$. By Lemma 2.1 we can verify that the following

$$F(a, U, f)_{\langle\langle b\rangle\rangle} = \begin{cases} f_{\langle\langle b\rangle\rangle} \cup U, & b = O_b \text{ and } a \gg b, \\ f_{\langle\langle b\rangle\rangle} \setminus U = f_{\langle\langle b\rangle\rangle}, & b = O_b \text{ and } a \notin \Uparrow b, \\ U, & b \neq O_b \text{ and } a \gg b, \\ \emptyset, & b \neq O_b \text{ and } a \notin \Uparrow b, \end{cases}$$

hold, i.e., $F(a, U, f)_{\langle\langle b\rangle\rangle} \in \Omega(X)$. By Theorem 1.1 $F(a, U, f) \in [X \to L]$. □

**Remark 2.3.** In this lemma, the condition $U \subseteq f_{\langle\langle a\rangle\rangle}$ is necessary. For example, let $L$ be the continuous L-Domain denoted by Fig. 1, and $X = L$ equipped with the Scott topology and $f$ the identity mapping on $L$. Take $U = \{b_1, b_2, b_3\} \in \Omega(X)$, then

$$F(a_3, U, f)(x) = \begin{cases} a_3, & x \in \{b_1, b_2, b_3\}, \\ O_a, & x \in \{a_1, a_2, a_3, O_a\}, \\ O_b, & x = O_b, \end{cases}$$

and hence $F(a_3, U, f)_{\langle\langle b\rangle\rangle} = \{O_b\} \notin \Omega(X)$ for $b = O_b$, i.e., $F(a_3, U, f) \notin [X \to L]$.

**Lemma 2.4.** *Let $L$ be a continuous L-Domain, $a, b \in L$ and $V \in \Omega(X)$, $f \in [X \to L]$. Then $F(b, V, f) \ll F(a, f_{\langle\langle a\rangle\rangle}, f)$ in $[X \to L]$ if $b \ll a$ and $V \ll f_{\langle\langle a\rangle\rangle}$.*

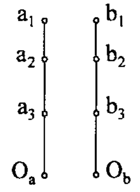


Fig. 1.

**Proof.** By Lemma 2.2, we have $F(b,V,f)$, $F(a,f_{\langle\langle a\rangle\rangle},f) \in [X \to L]$.

Now suppose that $\{h_j\colon j \in \Delta\}$ is a directed family in $[X \to L]$ with $\bigvee_{j\in\Delta} h_j \geqslant F(a,f_{\langle\langle a\rangle\rangle},f)$; then

$$\Big(\bigvee h_j\Big)_{\langle\langle b\rangle\rangle} = \bigcup_{j\in\Delta} h_{j\langle\langle b\rangle\rangle} \supseteq F(a,f_{\langle\langle a\rangle\rangle},f)_{\langle\langle b\rangle\rangle} \supseteq f_{\langle\langle a\rangle\rangle}$$

by the proof of Lemma 2.2. Hence we have $h_{j_0\langle\langle b\rangle\rangle} \supseteq V$ for some $j_0 \in \Delta$ because $\{h_{j\langle\langle b\rangle\rangle}\colon j \in \Delta\}$ is directed. By Lemma 2.1(2) $O_{f(x)} \leqslant h_{j_0}(x)$ for all $x \in X \setminus V$, hence $F(b,V,f) \leqslant h_{j_0}$. Thus the lemma is proved. $\square$

**Lemma 2.5.** *Let $L$ be an L-Domain, then we have the following conclusions:*

(1) *$L$ is continuous if and only if $\downarrow x$ is a continuous lattice for all $x \in L$.*

(2) *If $a,b \in \downarrow x \bigcap \downarrow y$ and $y \geqslant a \bigvee_x b = a \bigvee_y b$.*

**Proof.** (1) For $x \in L$ and $a \in \downarrow x$, $\Downarrow a \subseteq \Downarrow_x a = \{b \in \downarrow x\colon\ b \ll a \text{ in } \downarrow x\}$ is obvious and $\Downarrow_x a$ is directed because $\downarrow x$ is a complete lattice, which implies that $\downarrow x$ is a continuous lattice if $L$ is continuous. Conversely we need only show that $y \ll x$ in $\downarrow x$ implies $y \ll x$ in $L$ for all $x \in L$ and $y \in \downarrow x$. Suppose that $y \ll x$ in $\downarrow x$ and $D$ is a directed set in $L$ with $\bigvee D \geqslant x$ and let $e = \bigvee D$. Then $\Downarrow_e x = \{b \in \downarrow e\colon\ b \ll x \text{ in } \downarrow e\} \subseteq \downarrow x$ is directed and $\bigvee \Downarrow_e x = x$ as $\downarrow e$ is continuous, thus we have $y \leqslant b$ for some $b \in \Downarrow_e x$, i.e., $y \ll x$ in $\downarrow e$. As $D$ is also a directed set in $\downarrow e$ and $\bigvee D = e \geqslant x$, then $y \leqslant d$ for some $d \in D$, which implies $y \ll x$ in $L$.

(2) $a \bigvee_x b$ is obviously an upper bound of $\{a,b\}$ in $\downarrow y$ by $y \geqslant a \bigvee_x b$, hence $a \bigvee_x b \geqslant a \bigvee_y b$ holds. In this case we have $a \bigvee_y b \leqslant x$, which implies that $a \bigvee_y b$ is also an upper bound of $\{a,b\}$ in $\downarrow x$, so $a \bigvee_y b \geqslant a \bigvee_x b$ holds and hence $a \bigvee_x b = a \bigvee_y b$. $\square$

**Lemma 2.6.** *Let $L$ be a continuous L-Domain and $X$ a core compact space. Then $f = \bigvee \Downarrow f$ for all $f \in [X \to L]$.*

**Proof.** Note $F(a,f_{\langle\langle a\rangle\rangle},f) \leqslant f$ for all $a \in L$; it is easy to show

$$f = \bigvee_{a\in L} \bigvee_{b\in\Downarrow a} \bigvee_{V \ll f_{\langle\langle a\rangle\rangle}} F(b,V,f)$$

because both $L$ and $\Omega(X)$ are continuous. By Lemma 2.4 we obtain $f = \bigvee \Downarrow f$.

**Lemma 2.7.** *$[X \to L]$ is an L-Domain for each continuous L-Domain $L$ and core compact space $X$.*

**Proof.** Take an arbitrary $f \in [X \to L]$ and note that $\downarrow f$ is a DCPO; it is sufficient to show that $\downarrow f$ is a $\bigvee$-semilattice with least element. First of all we define the mapping $O_f : X \to L$ by $O_{f(x)} = O_f(x)$ for all $x \in X$, then $O_f$ is the least element of $\downarrow f$. Now take $g_1, g_2 \in \downarrow f$ and define the mapping $h : X \to L$ by $h(x) = g_1(x) \bigvee_{f(x)} g_2(x)$ for all $x \in X$, we claim $h = g_1 \bigvee_f g_2$.

Take an arbitrary $a \in L$ and $x \in h_{\langle\langle a \rangle\rangle}$, then one can find a element $\bar{a} \in L$ such that $h(x) \gg \bar{a} \gg a$. By Lemma 2.5(1) and $h(x) = \bigvee_{f(x)}(\Downarrow g_1(x) \cup \Downarrow g_2(x))$, there are finite subsets $\{a_1, a_2, \ldots, a_k\} \subseteq \Downarrow g_1(x)$, $\{b_1, b_2, \ldots, b_m\} \subseteq \Downarrow g_2(x)$ such that

$$h(x) \gg \bigvee\nolimits_{f(x)}\{a_i, b_j\colon\ i = 1, 2, \ldots, k;\ j = 1, 2, \ldots, m\} = e \geqslant \bar{a}.$$

Hence $f(x) \gg e$. Let

$$V = \bigcap_{i=1}^{k} g_{1\langle\langle a_i \rangle\rangle} \cap \bigcap_{i=1}^{k} g_{2\langle\langle b_j \rangle\rangle} \bigcap f_{\langle\langle e \rangle\rangle},$$

then $x \in V \in \Omega(X)$. For each $y \in V$, as

$$h(y) = g_1(y) \bigvee\nolimits_{f(y)} g_2(y) \gg \bigvee\nolimits_{f(y)}\{a_i, b_j\colon\ i = 1, 2, \ldots, k,\ j = 1, 2, \ldots, m\}$$

and $f(y) \gg e$, by Lemma 2.5(2)

$$e = \bigvee\nolimits_{f(y)}\{a_i, b_j\colon\ i = 1, 2, \ldots, k,\ j = 1, 2, \ldots, m\},$$

which implies $y \in h_{\langle\langle a \rangle\rangle}$, i.e., $V \subseteq h_{\langle\langle a \rangle\rangle}$. By arbitrariness of $x \in h_{\langle\langle a \rangle\rangle}$ we have $h_{\langle\langle a \rangle\rangle} \in \Omega(X)$ and hence $h \in [X \to L]$.

Now suppose $g \in \downarrow f$ such that $g_1 \leqslant g$ and $g_2 \leqslant g$, then $g(x) \geqslant g_1(x) \bigvee_{g(x)} g_2(x)$ for all $x \in X$. Again by Lemma 2.5(2) we have $g(x) \geqslant g_1(x) \bigvee_{f(x)} g_2(x) = h(x)$ as $f(x) \geqslant g(x)$, so $h = g_1 \bigvee_f g_2$ and the lemma is proved.

**Theorem 2.8.** *Let $L$ be a DCPO. Then $L$ is a continuous L-Domain if and only if $[X \to L]$ is a continuous L-Domain for all core compact spaces $X$.*

**Proof.** The necessity of the theorem follows from Lemmas 2.6 and 2.7. However, $L \cong [X \to L]$ if we take $X$ as one-point space, so the sufficiency of the theorem follows. □

From the proof of Lemma 2.7, the following corollary is obvious.

**Corollary 2.9.** *For a DCPO $L$, $L$ is bounded complete continuous if and only if $[X \to L]$ is bounded complete continuous for all core compact spaces $X$.*

In order to prove Theorem A, we need some notions. Following [3], we say that a DCPO $L$ has property $m$ if for each nonempty finite set $A \subseteq L$ and each upper bound $x_0$

of $A$, there is a minimal upper bound $x$ of $A$ such that $x \leqslant x_0$. A DCPO $L$ is bicomplete if each filtered set in $L$ has an infimum.

**Lemma 2.10.** *Each bicomplete DCPO $L$ has property $m$.*

**Proof.** Let $A$ be an nonempty finite set of $L$ and $x_0$ an upper bound of $A$, and write $\mu(A) = \{x \in \downarrow x_0\colon\ x \text{ is an upper bound of } A\}$ and $G(A) = \{B \subseteq \mu(A)\colon\ B \text{ is a filtered set}\}$; $G(A)$ is clearly a nonempty poset satisfying the condition of Zorn's Lemma as $\{x_0\} \in G(A)$. Hence $G(A)$ has a maximal element $B_0$ and $B_0$ has the infimum $b_0$ which is obviously a minimal upper bound of $A$ with $b_0 \leqslant x_0$. □

The following lemma appears as "Theorem 1.37" in [3], we omit the proof.

**Lemma 2.11** [3]. *A DCPO $L$ is bicomplete if $[L \to L]$ is continuous.*

For DCPOs $L$ and $L_1$, a continuous mapping $r : L \to L_1$ is called a retraction if there is a continuous mapping $h : L_1 \to L$ such that $r \circ h$ equals the identity mapping on $L_1$. In this case $L_1$ is called a retract of $L$ and $(r, h)$ a retraction embedding pair between $L$ and $L_1$.

**Lemma 2.12.** *Any retract of an L-Domain (continuous DCPO) is again an L-Domain (continuous DCPO), and $[X \to L_1]$ is a retract of $[X \to L]$ if $L_1$ is a retract of $L$ for all topological spaces $X$.*

**Proof.** Suppose that $L_1$ is a retract of $L$ and $(r, h)$ a retraction embedding pair between $L$ and $L_1$. Note that both $r$ and $h$ are order-preserving and $\downarrow x = \downarrow r \circ h(x) = r(\downarrow h(x))$ for each $x \in L_1$; the completeness of $\downarrow x$ follows from the completeness of $\downarrow h(x)$. Hence $L_1$ is an L-Domain if $L$ is. Also note that for each $x \in L_1$, $r(\Downarrow h(x)) = \{r(y)\colon\ y \in \Downarrow h(x)\} \subseteq \Downarrow x$; then $\Downarrow x$ is directed and $\bigvee \Downarrow x = x$ if $L$ is continuous. Now for each $f \in [X \to L]$ and $g \in [X \to L_1]$ define mappings $\bar{r} : [X \to L] \to [X \to L_1]$ and $\bar{h} : [X \to L_1] \to [X \to L]$ by $\bar{r}(f) = r \circ f$ and $\bar{h}(g) = h \circ g$; it is easy to verify that $(\bar{r}, \bar{h})$ is a retraction embedding pair between $[X \to L]$ and $[X \to L_1]$. Thus the lemma is proved. □

In [3] Jung gave a characterization of continuous L-Domains with least element in terms of retracts; we now give similarly a characterization of continuous L-Domains without least element.

**Lemma 2.13.** *Let $L$ be a continuous DCPO with continuous mapping space $[L \to L]$. Then $L$ is not an L-Domain if and only if $L$ contains $T_1$ or $M_1$ as a retract, where $T_1$ and $M_1$ are continuous DCPOs denoted by Fig. 2 and Fig. 3 respectively.*

**Proof.** It is obvious that an L-Domain can't contain $T_1$ or $M_1$ as a retract because neither $T_1$ nor $M_1$ is an L-Domain, hence the sufficiency of the lemma holds. Now suppose that $L$ is not an L-Domain. Then there is an $x \in L$ such that $\downarrow x$ has no least element or $\downarrow x$ contains a pair of elements $\{x_1, x_2\}$ without supremum in $\downarrow x$. We claim:

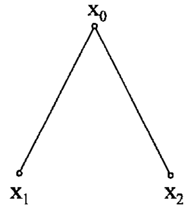


Fig. 2. $T_1$

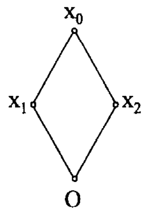


Fig. 2′.

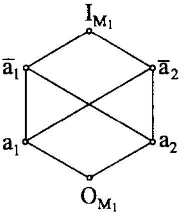


Fig. 3. $M_1$

(1) If $\downarrow x$ has no least element, then $L$ contains $T_1$ as a retract.

By Lemma 2.11 $\downarrow x$ has at least two minimal elements $\{\overline{x_1}, \overline{x_2}\}$ which are also minimal elements of $L$. Also by Lemma 2.10 and Lemma 2.11 $\{\overline{x_1}, \overline{x_2}\}$ has a minimum upper bound $x_0 \leqslant x$. Define a mapping $r : L \to \{x_0, \overline{x_1}, \overline{x_2}\}$ by

$$r(x) = \begin{cases} x_0, & x \notin \{\overline{x_1}, \overline{x_2}\}, \\ x, & x \in \{\overline{x_1}, \overline{x_2}\}; \end{cases}$$

it is not hard to see that $r$ is a retraction and its image $r(L) = \{x_0, \overline{x_1}, \overline{x_2}\}$ is a copy of $T_1$ inside $L$, so $L$ contains $T_1$ as a retract.

(2) If $\{x_1, x_2\}$ has no supremum in $\downarrow x$, then $L$ contains $M_1$ as a retract.

By (1) we can assume that $\downarrow x$ has the least element $O_x$, thus (2) follows from Jung's characterization [3, Proposition 4.22] and hence the lemma holds. □

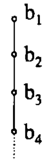


Fig. 4. $T_N$

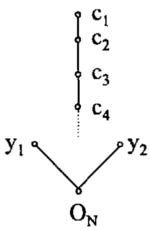


Fig. 5. $M_N$

**Lemma 2.14.** *Neither* $[T_N \to T_1]$ *nor* $[M_N \to M_1]$ *is continuous, where* $T_N$ *and* $M_N$ *are continuous DCPOs denoted by Fig.* 4 *and Fig.* 5 *respectively.*

**Proof.** Let $f_{x_1} \in [T_N \to T_1]$ be the constant mapping taking value $x_1$. It is not hard to see $f_{x_1} \notin \Downarrow f_{x_1}$; hence $\Downarrow f_{x_1} = \emptyset$ because $f_{x_1}$ is a minimal element of $[T_N \to T_1]$. Thus $[T_N \to T_1]$ is not continuous.

In order to show that $[M_N \to M_1]$ is not continuous, define mappings $f, g_1, g_2 : M_N \to M_1$ as follows:

$f(O_N) = O_{M_1}$, $f(y_1) = a_1$, $f(y_2) = a_2$ and $f(C_n) = 1_{M_1}$ for all $n \in \mathbb{N}$;

$g_1(O_N) = O_{M_1}$, $g_1(y_1) = a_1$, $g_1(y_2) = O_{M_1}$ and $g_1(C_n) = a_1$ for all $n \in \mathbb{N}$;

$g_2(O_N) = O_{M_1}$, $g_2(y_1) = O_{M_1}$, $g_2(y_2) = a_2$ and $g_2(C_n) = a_2$ for all $n \in \mathbb{N}$.

Obviously $f, g_1, g_2 : [M_N \to M_1]$. We can easily show $f \notin \Downarrow f$ and $g_1, g_2 \in \Downarrow f$. Now we claim that $\{g_1, g_2\}$ has no upper bound in $\Downarrow f$, i.e., $\Downarrow f$ is not directed and hence $[M_N \to M_1]$ is not continuous.

Let $h_1, h_2 \in [M_N \to M_1]$ be the mappings taking values $\bar{a}_1$ and $\bar{a}_2$ on the set $\{c_n\colon n \in \mathbb{N}\}$ respectively, $h_1(y_1) = h_2(y_1) = a_1$ and $h_1(y_2) = h_2(y_2) = a_2$. Obviously $h_i \notin \Downarrow f$ $(i = 1, 2)$. Note that if $g \leqslant f$ and $g$ is an upper bound of $\{g_1, g_2\}$, then $g \geqslant h_1$ or $g \geqslant h_2$ and hence $g \notin \Downarrow f$. Thus the claim holds and hence the lemma is proved. □

**Theorem 2.15** (Theorem A). *For DCPO* $L$, *the following conclusions are equivalent:*

(1) $L$ *is a continuous L-Domain;*

(2) $[X \to L]$ *is a continuous L-Domain for all core compact spaces* $X$;

(3) $[X \to L]$ *is a continuous DCPO for all core compact spaces* $X$.

**Proof.** (1) $\Rightarrow$ (2) follows from Theorem 2.8, (2) $\Rightarrow$ (3) is obvious, and we now prove (3) $\Rightarrow$ (1).

In fact $L$ is a continuous DCPO because $L \cong [X \to L]$ for the one-point space $X$. Now suppose that $L$ is not an L-Domain. By Lemma 2.13 $L$ contains $T_1$ or $M_1$ as a retract, and then by Lemma 2.12 $[X \to T_1]$ or $[X \to M_1]$ is a retract of $[X \to L]$ for all core compact spaces $X$. Especially, either $[T_N \to T_1]$ is a retract of $[T_N \to L]$ or $[M_N \to M_1]$ is a retract of $[M_N \to L]$. Again by Lemma 2.12 either $[T_N \to T_1]$ or $[M_N \to M_1]$ is a continuous DCPO, which contradicts Lemma 2.14. Hence (3) $\Rightarrow$ (1) holds. □

The following proposition is given by the referee.

**Proposition 2.16.** *$[X \to T_1]$ is a continuous DCPO for all spaces $X$ which are both compact and core compact, where $T_1$ is the continuous DCPO denoted by Fig. 2.*

**Proof.** Let $X$ be a core compact and compact space and consider the four-point lattice $L$ (given by Fig. 2′) with $T_1$ consisting of the top three points. Then $[X \to L]$ is a continuous lattice, hence a continuous DCPO. By the compactness of $X$, we can easily show that $[X \to T_1]$ is a Scott open subset of $[X \to L]$. Hence $[X \to T_1]$ is a continuous DCPO. □

Note that $T_1$ is not an L-Domain; this proposition shows that the second part of Problem A has a negative answer.

## 3. Solution to Problem B

The Isbell topology on $[X \to L]$, written $\mathrm{Is}(X, L)$, means the topology obtained by taking as a subbase for the open sets all sets of the form

$$N(H, V) = \{f \in [X \to L]\colon\ f^{-1}(V) \in H\},$$

where $H$ is a Scott open set in $\Omega(X)$ and $V$ is a Scott open set in $L$.

**Example 3.1.** Let $L$ be the DCPO denoted by Fig. 6. $L$ is obviously a bounded complete continuous DCPO without least element. Take $X = N$ equipped the discrete topology, then $X$ is a core compact space. We claim that the Isbell and Scott topologies on $[X \to L]$ do not agree.

In fact, by Theorem 2.8 $[X \to L]$ is a continuous L-Domain and hence $\{\Uparrow f\colon\ f \in [X \to L]\}$ is a base of the Scott topology on $[X \to L]$. Define mappings $f, g : X \to L$ as follows:

$$g(1) = a_1,\ g(2) = a_2 \text{ and } g(n) = a_3 \quad \text{for all } n \geqslant 3;$$

$$f(1) = a_1,\ f(2) = O_2 \text{ and } f(n) = O_3 \quad \text{for all } n \geqslant 3,$$

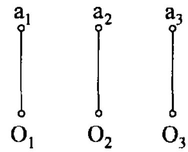


Fig. 6.

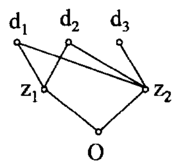


Fig. 7. $L(N)$

obviously $f, g \in [X \to L]$ and $g \in \Uparrow f$. Also note that

$$\{\uparrow F = \{B \subseteq N\colon\ B \supseteq F\}\colon\ F \text{ is a finite subset of } N\}$$

is a base of the Scott topology $\sigma(\Omega(X))$; we can show that

$$B(g) = \Big\{N(\uparrow\{1\}, \{a_1\}) \cap N(\uparrow\{2\}, \{a_2\}) \cap N(\uparrow F, \{a_3\})\colon \\ F \text{ is a finite subset of } N \setminus \{1,2\}\Big\}$$

is an Isbell neighbourhood base of $g$. Take an arbitrary

$$V = N(\uparrow\{1\}, \{a_1\}) \cap N(\uparrow\{2\}, \{a_2\}) \cap N(\uparrow F, \{a_3\}) \in B(g)$$

and let $j_0 = \max F$, define a mapping $h\colon N \to L$ by $h(1) = a_1$, $h(2) = a_2$, $h(j_0 + 1) = O_2$ and $h(n) = a_3$ for all $n \in \mathbb{N} \setminus \{1, 2, j_0 + 1\}$. It is easy to see $h \in V$ but $h \notin \Uparrow f$ as $h(j_0 + 1) = O_2 \not\geqslant f(j_0 + 1) = O_3$. So $V \not\subseteq \Uparrow f$ and hence by arbitrariness of $V \Uparrow f$ is not Isbell open, i.e., $\mathrm{Is}(X, L) \neq \sigma([X \to L])$.

**Example 3.2.** Let $L(N)$ be the continuous L-Domain with least element $O$ denoted by Fig. 7 and $X = (L(N), \sigma(L(N)))$. By Theorem 2.8 $[X \to L(N)]$ is a continuous L-Domain with least element.

Note that both $L$ and $\Omega(X)$ are countably based continuous DCPO, so $\mathrm{Is}(X, L)$ has a countable base. However $\sigma([X \to L])$ doesn't have a countable base and hence $\mathrm{Is}(X, L) \neq \sigma([X \to L])$.

From the definition of bounded complete DCPO, the following lemma is clear.

**Lemma 3.3.** *A DCPO $L$ is bounded complete if and only if each pair of elements with upper bound has a supremum.*

**Lemma 3.4.** *Let $L$ be a continuous L-Domain with least element $O_L$. Then $L$ is a bounded complete DCPO if* $\mathrm{Is}(X,L) = \sigma([X \to L])$ *for all core compact spaces $X$.*

**Proof.** Suppose that $L$ is not bounded complete. By Lemma 3.3 there is a pair of elements $\{b_1, b_2\}$ with upper bound which has no supremum, which implies $\{b_1, b_2\}$ has at least two minimal upper bounds $\{a_1, a_2\}$ and $\uparrow a_1 \cap \uparrow a_2 = \emptyset$ because $L$ is an L-Domain. By Theorem 1.1 and continuity of $L$ we can find the elements $b_1'$ and $b_2'$ with $b_1' \ll b_1$ and $b_2' \ll b_2$ such that $\{b_1', b_2'\}$ has no supremum, then both $a_1^* = b_1' \bigvee_{a_1} b_2'$ and $a_2^* = b_1' \bigvee_{a_2} b_2'$ are minimal upper bounds of $\{b_1', b_2'\}$ and $\uparrow a_1^* \cap \uparrow a_2^* = \emptyset$. Now take $X = L(N)$ (denoted by Fig. 7) equipped with the Scott topology, and define mappings $f, g : X \to L$ as follows:

$$f(O) = O_L,\ f(z_1) = b_1',\ f(z_2) = b_2' \text{ and } f(d_n) = \begin{cases} a_1^*, & n \text{ an odd number}, \\ a_2^*, & n \text{ an even number}, \end{cases}$$

$$g(O) = O_L,\ g(z_1) = b_1,\ g(z_2) = b_2 \text{ and } g(d_n) = \begin{cases} a_1, & n \text{ an odd number}, \\ a_2, & n \text{ an even number}, \end{cases}$$

we can verify that $g, f \in [X \to L]$ and $f \ll g$. By Theorem 2.8 $\Uparrow f$ is a Scott open neighbourhood of $g$. Write $\mathcal{F}_1$ and $\mathcal{F}_2$ for the sets of all nonempty finite subsets of $\{x_{2n-1}\colon n \in \mathbb{N}\}$ and $\{x_{2n}\colon n \in \mathbb{N}\}$ respectively, and let

$$\begin{aligned} B(g) = \Big\{ & N(\uparrow F_1, \Uparrow t_1) \cap N(\uparrow F_2, \Uparrow t_2) \cap N(\uparrow\uparrow z_1, \Uparrow t_3) \cap N(\uparrow\uparrow z_2, \Uparrow t_4): \\ & F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2 \text{ and } a_1^* \leqslant t_1 \ll a_1,\ a_2^* \leqslant t_2 \ll a_2, \\ & t_3 \in \Downarrow b_1 \setminus \downarrow b_2,\ t_4 \in \Downarrow b_2 \setminus \downarrow b_1 \Big\}. \end{aligned}$$

Then $B(g)$ is an Isbell neighbourhood base of $g$. Take an arbitrary

$$V = N(\uparrow F_1, \Uparrow t_1) \cap N(\uparrow F_2, \Uparrow t_2) \cap N(\uparrow\uparrow z_1, \Uparrow t_3) \cap N(\uparrow\uparrow z_2, \Uparrow t_4) \in B(g)$$

and let $j_0 = \max\{i \in \mathbb{N}\colon x_i \in F_1 \cup F_2\}$, and define mapping $h\colon X \to L$ as follows:

$$h(x) = g(x) \quad \text{for all } x \in L(N) \setminus \{d_{2j_0+1}\} \text{ and } h(d_{2j_0+1}) = a_2.$$

It is not hard to see $h \in [X \to L]$ and $h \in V$, but $h \notin \Uparrow f$ as $h(x_{2j_0+1}) = a_2 \not\geqslant a_1^*$ (otherwise $a_2 \in \uparrow a_1^* \cap \uparrow a_1^* = \emptyset$). Hence $V \not\subseteq \Uparrow f$, which implies that $\Uparrow f$ is not an Isbell open set, so $\mathrm{Is}(X, L) \neq \sigma([X \to L])$. This is a contradiction and hence the lemma is proved. □

**Theorem 3.5** (Theorem B). *Let $L$ be a continuous L-Domain with least element $O_L$. Then* $\mathrm{Is}(X,L) = \sigma([X \to L])$ *for all core compact spaces $X$ if and only if $L$ is a bounded complete DCPO.*

**Proof.** The necessity of the theorem follows from Lemma 3.4. Conversely, note that $\mathrm{Is}(X, L) \subseteq \sigma([X \to L])$. By Theorem 1.1 we need only show that $\Uparrow f$ is an Isbell open set for each $f \in [X \to L]$. Write $bV$ for the mapping taking value $b$ on $V$ and value $O_L$ otherwise, and take an arbitrary $f \in [X \to L]$.

For each $g \in \Uparrow f$, by Lemma 2.6 we have

$$g = \bigvee_{a \in L} \bigvee_{b \in \Downarrow a} \bigvee_{V \in \Downarrow g_{\langle\langle a \rangle\rangle}} F(b, V, g) = \bigvee_{a \in L} \bigvee_{b \in \Downarrow a} \bigvee_{V \in \Downarrow g_{\langle\langle a \rangle\rangle}} bV.$$

Then by Corollary 2.9 there exists a finite subset $\{b_1, b_2, \ldots, b_k\} \subseteq L$ such that $f \leqslant \bigvee\{b_i V_i \colon i = 1, 2, 3, \ldots, k\}$, where $b_i \in \Downarrow a_i$ and $V_i \in \Downarrow g_{\langle\langle a_i \rangle\rangle}$ $(\forall i = 1, 2, \ldots, k)$. Obviously $H = \bigcap_{i=1}^{k} N(\Uparrow V_i, \Uparrow a_i)$ is an Isbell neighbourhood of $g$, we claim $H \subseteq \Uparrow f$:

In fact, for each $h \in H$, by Corollary 2.9 and Lemma 2.4 we have

$$f \leqslant \bigvee\{b_i V_i \colon i = 1, 2, \ldots, k\} \ll \bigvee\{a_i h_{\langle\langle a \rangle\rangle} \colon i = 1, 2, \ldots, k\} \leqslant h$$

because $V_i \ll h_{\langle\langle a_i \rangle\rangle}$ and $b_i \ll a_i$ $(i = 1, 2, \ldots, k)$, so $h \in \Uparrow f$. By the arbitrariness of $g \in \Uparrow f$, it follows that $\Uparrow f$ is Isbell open and thus the theorem is proved. $\square$

By Examples 3.1 and 3.2 the condition that $L$ is a continuous L-Domain with least element in Theorem 3.5 is suitable.

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